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GENERALIZED DUNDURS CONSTANTS FOR ANISOTROPIC BIMATERIALS

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Abstract—It is well known that when a homogeneous isotropic elastic medium is under a plane strain deformation due to prescribed tractions at the boundary, the stress is independent of elastic constants. In the case of a bimaterial that consists of two dissimilar isotropic materials bonded together along their interface, the stress depends on two composite elastic constants known as Dundurs constants. For anisotropic elastic materials, plane strain deformations are possible for monoclinic materials with the symmetry plane at $x_3 = 0$. For these materials, the plane strain solution is in terms of two complex variables $z_1 = x_1 + p_1 x_2$ and $z_2 = x_1 + p_2 x_2$, where p_1 and p_2 are complex eigenvalues depending on elastic constants. If the boundary conditions are prescribed in terms of tractions, it is shown that the stress is independent of elastic constants except p_1 and p_2. In the case of a bimaterial that consists of two dissimilar monoclinic materials bonded together along their interface, the stress depends on two composite elastic constants α and β (in addition to p_1 and p_2 in both materials). They reduce to Dundurs constants in the isotropic limit. The result remains valid when the interface is not perfectly bonded. We also show that every plane strain solution to a given anisotropic material or bimaterial is applicable to a wider class of anisotropic materials and bimaterials. In particular, every plane strain solution to an isotropic material or bimaterial is applicable to a class of anisotropic materials or bimaterials that may possess no material symmetry. Finally, we show how the results obtained here can be modified for plane stress deformations.

1. INTRODUCTION

Consider a bimaterial that consists of two dissimilar homogeneous isotropic media bonded together along their interface. The geometry of the interface can be rather arbitrary. There are two elastic constants each in the two materials, resulting in a total of four elastic constants. The solution for the stress depends on these four elastic constants, but they can be reduced to three by a dimensional analysis. Dundurs (1969a,b, 1970) has proved that the solution for the stress depends on two composite elastic constants (known as Dundurs constants) provided:

(i) the deformation is plane strain, i.e. u_1, u_2 depend on x_1, x_2 only while $u_3 = 0$;
(ii) the boundary conditions are prescribed in terms of tractions;
(iii) the Michell (1899) condition is satisfied, i.e. the integration of the surface tractions on any closed curve vanishes.

Condition (i) is not possible for general anisotropic materials because the in-plane displacement (u_1, u_2) and the anti-plane displacement u_3 are in general coupled. For anisotropic elastic materials, the stress–strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks} u_{k,s} \tag{1}$$

$$C_{ijks} u_{k,sj} = 0, \tag{2}$$

in which u_i, σ_{ij} are the displacement and stress, respectively, a comma denotes differentiation, repeated indices imply summation, and C_{ijks} are the elastic stiffnesses assumed to possess the full symmetry. Equation (2) consists of three equations. With condition (i) the third equation vanishes identically if

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{46} = C_{56} = 0, \tag{3}$$

where we have employed the contracted notation for C_{ijks} (Lekhnitskii, 1950). Equation (3) is satisfied by monoclinic materials with the symmetry plane at $x_3 = 0$. In fact eqn (3) represents materials more general than monoclinic materials with the symmetry plane at $x_3 = 0$ because the latter also require that $C_{34} = C_{35} = 0$. Since C_{34} and C_{35} are not needed in plane strain deformations, we will consider in this paper monoclinic materials with the symmetry plane at $x_3 = 0$. The anti-plane displacement u_3 is uncoupled from (u_1, u_2), allowing us to set $u_3 = 0$. Only six out of thirteen elastic constants are needed for these materials under a plane strain deformation. This means a total of twelve elastic constants for a monoclinic bimaterial. More discussions on decoupling of in-plane and anti-plane deformations are given by Horgan and Miller (1994) and Ting (1994b).

The general solution to eqns (1) and (2) is presented in section 2. In section 3 we specialize the solution to plane strain deformations in monoclinic materials with the symmetry plane at $x_3 = 0$. The general solution is a superposition of functions of complex variables $z_1 = x_1 + p_1x_2$ and $z_2 = x_1 + p_2x_2$, where p_1 and p_2 are complex eigenvalues depending on elastic constants. The real and imaginary parts of p_1, p_2 account for four elastic constants. It is shown that the solution for the stress in a homogeneous material subjected to conditions (ii) and (iii) does not depend on elastic constants other than p_1 and p_2. Bimaterials that consist of two dissimilar monoclinic materials bonded together along their interface are considered in Section 4. If conditions (ii) and (iii) are satisfied, the solution for the stress depends on two 2×2 real tensors $\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$. We prove in Section 5 that $\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$ depend on two composite elastic constants α and β in addition to p_1, p_2 in the two materials. The p_1, p_2 in the two materials account for eight elastic constants for the bimaterial. The α and β reduce to Dundurs constants when the materials are isotropic. It is shown in section 6 that the stress remains dependent on α and β also for an interface that may consist of cracks and/or a sliding interface with or without friction. Several special monoclinic materials for which $\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$ have a simple expression are presented in section 7. In section 8 we point out by an example that any plane strain solution to an anisotropic elastic material or bimaterial is applicable to a wider class of anisotropic elastic materials or bimaterials. The last section shows that the results presented can be applied to plane stress deformations with a simple modification.

2. GENERAL SOLUTION

A general solution to eqn (2) is (Eshelby *et al.*, 1953)

$$u_i = a_i f(z) \quad \text{or} \quad \mathbf{u} = \mathbf{a} f(z) \tag{4}$$

where

$$z = x_1 + p x_2.$$

In the above, f is an arbitrary function of z, and p and a_i are determined by inserting eqn (4) into eqn (2). In matrix notation we have

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}\}\mathbf{a} = \mathbf{0}, \tag{5}$$

where the superscript T denotes the transpose and the 3×3 matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{6}$$

The matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite if the strain energy is positive.

Introducing the new vector

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \tag{7}$$

in which the second equality follows from eqn (5), the stress obtained from substituting eqn (4) into eqn (1) can be written in terms of the stress function $\boldsymbol{\phi}$ as (Stroh, 1958, 1962)

$$\sigma_{i1} = -\varphi_{i,2}, \quad \sigma_{i2} = \varphi_{i,1} \tag{8}$$

$$\varphi_i = b_i f(z) \quad \text{or} \quad \boldsymbol{\phi} = \mathbf{b} f(z). \tag{9}$$

It is sufficient, therefore, to consider the stress function $\boldsymbol{\phi}$ because the stress can be obtained from eqn (8) by differentiation. Let $\mathbf{t}$ be the surface traction at a boundary surface. Then

$$\mathbf{t} = \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\phi},$$

where s is the arclength of the boundary (Stroh, 1958). This is reduced to eqn (8) when the boundary is a plane parallel to the x_2-axis or the x_1-axis. Hence

$$\boldsymbol{\phi} = \int \mathbf{t}\,\mathrm{d}s + \boldsymbol{\phi}^{\mathrm{o}}, \tag{10}$$

where $\boldsymbol{\phi}^{\mathrm{o}}$ is a constant vector. If the Michell condition is satisfied, the integral in eqn (10) vanishes when the integration is taken around any complete circle.

Equation (5) provides six eigenvalues for p and six associate eigenvectors $\mathbf{a}$. Since p cannot be real if the strain energy is positive, there are three pairs of complex conjugates for p. If p_κ ($\kappa = 1, 2, \ldots, 6$) are the eigenvalues we let

$$\operatorname{Im} p_\kappa > 0, \quad p_{\kappa+3} = \bar{p}_\kappa, \quad \mathbf{a}_{\kappa+3} = \bar{\mathbf{a}}_\kappa, \quad \mathbf{b}_{\kappa+3} = \bar{\mathbf{b}}_\kappa \quad (\kappa = 1, 2, 3), \tag{11}$$

where Im stands for the imaginary part and the overbar denotes the complex conjugate. Assuming that the p_κ are distinct, the general solutions obtained by superposing six solutions of the form in eqns (4) and (9) are

$$\mathbf{u} = \sum_{\kappa=1}^{3} \{\mathbf{a}_\kappa f_\kappa(z_\kappa) + \bar{\mathbf{a}}_\kappa g_\kappa(\bar{z}_\kappa)\}, \quad \boldsymbol{\phi} = \sum_{\kappa=1}^{3} \{\mathbf{b}_\kappa f_\kappa(z_\kappa) + \bar{\mathbf{b}}_\kappa g_\kappa(\bar{z}_\kappa)\}, \tag{12}$$

where $f_\kappa, g_\kappa (\kappa = 1, 2, 3)$ are arbitrary functions of their arguments and

$$z_\kappa = x_1 + p_\kappa x_2. \tag{13}$$

In most applications $g_\kappa = \bar{f}_\kappa$ so that $\mathbf{u}$ and $\boldsymbol{\phi}$ are real. An exception is the stress singularity analysis in which $f_\kappa = z_\kappa^{\delta+1}$ and $g_\kappa = \bar{z}_\kappa^{\delta+1}$, where δ is the order of stress singularity (Wang and Choi, 1982). If δ is not real, $g_\kappa \neq \bar{f}_\kappa$ and $\mathbf{u}$ and $\boldsymbol{\phi}$ may not be real. However, if δ is a stress singularity so is $\bar{\delta}$ (Ting, 1986). One can therefore superimpose two solutions associated with δ and $\bar{\delta}$ to obtain real $\mathbf{u}$ and $\boldsymbol{\phi}$.

It is more convenient to write eqn (12) as

$$\mathbf{u} = \mathbf{A}\mathbf{f}(z) + \bar{\mathbf{A}}\mathbf{g}(\bar{z}), \quad \boldsymbol{\phi} = \mathbf{B}\mathbf{f}(z) + \bar{\mathbf{B}}\mathbf{g}(\bar{z}), \tag{14}$$

where $\mathbf{A}$ and $\mathbf{B}$ are 3×3 complex matrices given by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3],$$

and

$$\mathbf{f}(z) = \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{bmatrix}, \quad \mathbf{g}(z) = \begin{bmatrix} g_1(\bar{z}_1) \\ g_2(\bar{z}_2) \\ g_3(\bar{z}_3) \end{bmatrix}.$$

When the vectors $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$ are properly normalized, the three Barnett–Lothe tensors **S, H, L**, defined by

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^{\mathrm{T}}, \tag{15}$$

are real (Barnett and Lothe, 1973; Chadwick and Smith, 1977). **H** and **L** are symmetric and positive definite, while $\mathbf{SL}^{-1}$ and $\mathbf{H}^{-1}\mathbf{S}$ are skew-symmetric. The impedance tensor **M** and its inverse, defined by (Ingebrigtsen and Tonning, 1969)

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1} = \mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S}, \quad \mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}, \tag{16}$$

are positive definite Hermitian. With eqn (16), we write eqn (14) as

$$\mathbf{u} = -i\mathbf{M}^{-1}\mathbf{v}(z) + i\bar{\mathbf{M}}^{-1}\mathbf{w}(\bar{z}), \quad \boldsymbol{\phi} = \mathbf{v}(z) + \mathbf{w}(\bar{z}), \tag{17}$$

where

$$\mathbf{v}(z) = \mathbf{B}\mathbf{f}(z), \quad \mathbf{w}(\bar{z}) = \bar{\mathbf{B}}\mathbf{g}(\bar{z}). \tag{18}$$

3. MONOCLINIC MATERIALS WITH THE SYMMETRY PLANE AT $x_3 = 0$

For monoclinic materials with the symmetry plane at $x_3 = 0$, the matrices **A, B** and $\mathbf{M}^{-1}$ in eqns (14) and (17) have the structure (Suo, 1990; Ting, 1992a)

$$\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix},$$

where the * denotes a possibly nonzero element. By setting $f_3(z_3) = g_3(\bar{z}_3) = 0$, u_3 and φ_3 vanish identically. We therefore have a plane strain deformation. The third rows and third columns of **A, B** and $\mathbf{M}^{-1}$ in eqns (14) and (17) can be deleted, and **A, B** and $\mathbf{M}^{-1}$ are reduced to 2×2 matrices. $\mathbf{u}$, $\boldsymbol{\phi}$, $\mathbf{v}$ and $\mathbf{w}$ are 2×1 column matrices. The general solution given by eqn (17) is in terms of two complex variables, $z_1 = x_1 + p_1x_2$ and $z_2 = x_1 + p_2x_2$.

The eigenvalues p_1 and p_2 for monoclinic materials with the symmetry plane at $x_3 = 0$ can be obtained from eqn (5). An alternate equation is (Lekhnitskii, 1950)

$$s'_{11}p^4 - 2s'_{16}p^3 + (2s'_{12} + s'_{66})p^2 - 2s'_{26}p + s'_{22} = 0, \tag{19}$$

in which s'_{ij} are the reduced elastic compliances

$$s'_{ij} = s_{ij} - \frac{s_{i3}s_{3j}}{s_{33}}, \tag{20}$$

and s_{ij} are the elastic compliances. They are related to the elastic stiffnesses C_{ij} by

$$C_{ij}s_{jk} = \delta_{ik},$$

where δ_{ik} is the Kronecker delta. p_1, p_2 are the roots of eqn (19) with positive imaginary parts. Explicit expressions of **A** and **B** are (Suo, 1990; Ting, 1992a)

$$\mathbf{A} = \begin{bmatrix} \xi(p_1) & \xi(p_2) \\ \eta(p_1) & \eta(p_2) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -p_1 & -p_2 \\ 1 & 1 \end{bmatrix}, \tag{21}$$

in which

$$\xi(p) = s'_{11}p^2 - s'_{16}p + s'_{12}, \quad \eta(p) = s'_{12}p - s'_{26} + s'_{22}p^{-1}.$$

The columns of **A** and **B** must be normalized according to

$$2\mathbf{a}_1 \cdot \mathbf{b}_1 = 1, \quad 2\mathbf{a}_2 \cdot \mathbf{b}_2 = 1,$$

in computing **S, H,** and **L** in eqn (15). Normalization is not required for the general solution (14), the impedance tensor **M** and its inverse $\mathbf{M}^{-1}$ in eqn (16) [see Ting (1992a)], and the general solution (17).

From eqns (19) and (21), the matrix **A** depends on the six elastic compliances s'_{11}, s'_{12}, s'_{22}, s'_{16}, s'_{26}, and s'_{66}. However **B** depends on p_1, p_2 only. If the boundary conditions are prescribed in terms of tractions and the Michell condition is satisfied, $\boldsymbol{\phi}$ at the boundary, determined from eqn (10), is single-valued and is independent of elastic constants. Consequently, the unknown functions $\mathbf{v}(z)$ and $\mathbf{w}(\bar{z})$ in eqn $(17)_2$, determined by the boundary conditions, do not depend on elastic constants. We conclude, therefore, that when the boundary conditions are prescribed in terms of tractions and the Michell condition is satisfied, the stress does not depend on elastic constants other than p_1 and p_2. For isotropic materials $p_1 = p_2 = i$ and, as expected, the stress does not depend on elastic constants for plane strain deformations of isotropic materials when the boundary conditions are prescribed in terms of tractions.

It should be noted that the boundary conditions for a concentrated force are excluded because they are not prescribed entirely in terms of tractions. The displacement must be single-valued when one goes around the concentrated force one complete circle. A concentrated force does not satisfy the Michell condition. It demands that $\boldsymbol{\phi}$, and hence **v** and **w**, be multi-valued. This implies that the displacement **u**, obtained from eqn $(17)_1$, may be multi-valued, requiring the imposition of continuity of displacement. The dependence on elastic constants of isotropic materials when the Michell condition is violated was studied by Dundurs (1967a).

4. MONOCLINIC BIMATERIALS

Consider now a bimaterial that consists of two dissimilar monoclinic materials. Let the solution be

$$\mathbf{u}_1 = -i\mathbf{M}_1^{-1}\mathbf{v}_1(z^{(1)}) + i\bar{\mathbf{M}}_1^{-1}\mathbf{w}_1(\bar{z}^{(1)}), \quad \boldsymbol{\phi}_1 = \mathbf{v}_1(z^{(1)}) + \mathbf{w}_1(\bar{z}^{(1)}), \tag{22}$$

for material 1 and

$$\mathbf{u}_2 = -i\mathbf{M}_2^{-1}\mathbf{v}_2(z^{(2)}) + i\bar{\mathbf{M}}_2^{-1}\mathbf{w}_2(\bar{z}^{(2)}), \quad \boldsymbol{\phi}_2 = \mathbf{v}_2(z^{(2)}) + \mathbf{w}_2(\bar{z}^{(2)}), \tag{23}$$

for material 2. The continuity of displacement and traction at the interface means that, by eqn (10),

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\phi}_1 - \boldsymbol{\phi}_1^{\circ} = \boldsymbol{\phi}_2 - \boldsymbol{\phi}_2^{\circ}.$$

When condition (iii) is satisfied, $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are single-valued. We may set $\boldsymbol{\phi}_1^{\circ} = \boldsymbol{\phi}_2^{\circ}$. Substitution of eqns (22) and (23) into the above leads to

$$\mathbf{M}_1^{-1}\mathbf{v}_1-\bar{\mathbf{M}}_1^{-1}\mathbf{w}_1=\mathbf{M}_2^{-1}\mathbf{v}_2-\bar{\mathbf{M}}_2^{-1}\mathbf{w}_2,\quad \mathbf{v}_1+\mathbf{w}_1=\mathbf{v}_2+\mathbf{w}_2. \tag{24}$$

For simplicity we have omitted the arguments of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$. Equations (24) are the continuity conditions at the interface in which elastic constants appear in eqn $(24)_1$ through $\mathbf{M}_1$ and $\mathbf{M}_2$. With the external load prescribed in terms of tractions, $(24)_1$ is the only place elastic constants enter into the unknown functions $\mathbf{v}_n$, $\mathbf{w}_n$ $(n=1,2)$. Dependence on $p_1^{(n)}, p_2^{(n)}$ $(n=1,2)$ is understood.

Inserting $\mathbf{w}_2$ from eqn $(24)_2$ into eqn $(24)_1$ we obtain

$$(\mathbf{M}_1^{-1}+\bar{\mathbf{M}}_2^{-1})\mathbf{v}_1-(\bar{\mathbf{M}}_1^{-1}-\bar{\mathbf{M}}_2^{-1})\mathbf{w}_1=(\mathbf{M}_2^{-1}+\bar{\mathbf{M}}_2^{-1})\mathbf{v}_2.$$

From eqns $(16)_{3,4}$ this can be written as

$$(\mathbf{D}-i\mathbf{W})\mathbf{v}_1-(\mathbf{U}+i\mathbf{W})\mathbf{w}_1=(\mathbf{D}-\mathbf{U})\mathbf{v}_2$$

or

$$\mathbf{D}(\mathbf{v}_1-\mathbf{v}_2)+\mathbf{U}(\mathbf{v}_2-\mathbf{w}_1)-i\mathbf{W}(\mathbf{v}_1+\mathbf{w}_1)=\mathbf{0}. \tag{25}$$

In the above

$$\mathbf{D}=\mathbf{L}_1^{-1}+\mathbf{L}_2^{-1},\quad \mathbf{U}=\mathbf{L}_1^{-1}-\mathbf{L}_2^{-1},\quad \mathbf{W}=\mathbf{S}_1\mathbf{L}_1^{-1}-\mathbf{S}_2\mathbf{L}_2^{-1}, \tag{26}$$

are tensors. Premultiplied by $\mathbf{D}^{-1}$, eqn (25) takes the form

$$(\mathbf{v}_1-\mathbf{v}_2)+(\mathbf{D}^{-1}\mathbf{U})(\mathbf{v}_2-\mathbf{w}_1)-i(\mathbf{D}^{-1}\mathbf{W})(\mathbf{v}_1+\mathbf{w}_1)=\mathbf{0}. \tag{27}$$

We arrive at the result that the solution for the stress depends on the two 2×2 real tensors

$$\mathbf{D}^{-1}\mathbf{U} \quad \text{and} \quad \mathbf{D}^{-1}\mathbf{W}. \tag{28}$$

In the next section we show that $\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$ depend on two composite elastic constants α and β in addition to $p_1^{(n)}$, $p_2^{(n)}$ $(n=1,2)$.

It should be noted that when $\mathbf{w}_1=\mathbf{v}_2$ at the interface, eqn (27) shows that the stress solution depends on $\mathbf{D}^{-1}\mathbf{W}$ only. This is the case for an interface crack in a bimaterial subjected to prescribed tractions on the crack surfaces (Qu and Li, 1991; Ting, 1992b). If, in addition, $\mathbf{D}^{-1}\mathbf{W}=\mathbf{0}$, the solution for the stress is independent of elastic constants except p_1 and p_2 in the two materials. Bimaterials for which $\mathbf{D}^{-1}\mathbf{W}\neq\mathbf{0}$ are called *mismatched* bimaterials.

5. GENERALIZED DUNDURS CONSTANTS FOR ANISOTROPIC BIMATERIALS

When $\mathbf{A}$ and $\mathbf{B}$ of eqn (21) are inserted into eqn $(16)_4$ and use is made of eqn (19), $\mathbf{L}^{-1}$ and $\mathbf{SL}^{-1}$ can be obtained explicitly (Suo, 1990; Ting, 1992a). We write the result in the form

$$\mathbf{L}^{-1}=\upsilon\mathbf{P},\quad \mathbf{SL}^{-1}=\omega\mathbf{J}, \tag{29}$$

where

$$\mathbf{P}=\frac{1}{\gamma}\begin{bmatrix} b & d \\ d & e \end{bmatrix},\quad \mathbf{J}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\quad \mathbf{JJ}=-\mathbf{I} \tag{30}$$

$$\gamma>0,\quad \upsilon=\gamma s'_{11}>0,\quad \omega=s'_{12}-s'_{11}\mathrm{Re}\,(p_1p_2)>0 \tag{31}$$

$$b = \mathrm{Im}(p_1+p_2) > 0, \quad d = \mathrm{Im}(p_1 p_2), \quad e = \mathrm{Im}\{p_1 p_2(\bar{p}_1+\bar{p}_2)\} > 0. \tag{32}$$

In the above, Re and Im stand for the real and imaginary parts, respectively. We have introduced a positive scaling factor γ. It is a function of p_1, p_2 to be determined later in such a way that υ is an invariant. The elastic constants appear explicitly only in υ and ω. **J** is a constant matrix while **P** depends on p_1 and p_2 only. **P** is positive definite because $\mathbf{L}^{-1}$ is. Hence $b > 0$ and $e > 0$ as shown in eqn (32). We also have $be-d^2 > 0$. The inequalities $\upsilon > 0$, $\omega > 0$ (Ting, 1992a) in eqn (31) hold when the strain energy is strictly positive definite. This means that the strain energy is nonzero whenever the stress or strain is. This is not the case if the material is incompressible or if an elastic stiffness is infinity because the strain energy can be zero with a nonzero stress. In that case, the inequalities for υ and ω in eqn (31) are replaced by $\upsilon \geqslant 0$, $\omega \geqslant 0$.

From eqns (26) and (29) we have

$$\begin{aligned} \mathbf{D} = (\upsilon_1 \mathbf{P}_1+\upsilon_2 \mathbf{P}_2) &= \tfrac{1}{2}(\upsilon_1+\upsilon_2)(\mathbf{P}_1+\mathbf{P}_2)+\tfrac{1}{2}(\upsilon_1-\upsilon_2)(\mathbf{P}_1-\mathbf{P}_2) \\ &= \tfrac{1}{2}(\upsilon_1+\upsilon_2)\mathbf{X} \end{aligned} \tag{33}$$

$$\begin{aligned} \mathbf{U} = (\upsilon_1 \mathbf{P}_1-\upsilon_2 \mathbf{P}_2) &= \tfrac{1}{2}(\upsilon_1+\upsilon_2)(\mathbf{P}_1-\mathbf{P}_2)+\tfrac{1}{2}(\upsilon_1-\upsilon_2)(\mathbf{P}_1+\mathbf{P}_2) \\ &= \tfrac{1}{2}(\upsilon_1+\upsilon_2)\mathbf{Y} \end{aligned} \tag{34}$$

$$\mathbf{W} = (\omega_1-\omega_2)\mathbf{J} = (\upsilon_1+\upsilon_2)\beta\mathbf{J}, \tag{35}$$

where

$$\mathbf{X} = (\mathbf{P}_1+\mathbf{P}_2)+\alpha(\mathbf{P}_1-\mathbf{P}_2), \quad \mathbf{Y} = (\mathbf{P}_1-\mathbf{P}_2)+\alpha(\mathbf{P}_1+\mathbf{P}_2) \tag{36}$$

$$\alpha = \frac{\upsilon_1-\upsilon_2}{\upsilon_1+\upsilon_2}, \quad \beta = \frac{\omega_1-\omega_2}{\upsilon_1+\upsilon_2}. \tag{37}$$

X and **Y** depend on α, not on β. Thus

$$\mathbf{D}^{-1}\mathbf{U} = \mathbf{X}^{-1}\mathbf{Y}, \quad \mathbf{D}^{-1}\mathbf{W} = 2\beta\mathbf{X}^{-1}\mathbf{J}. \tag{38}$$

They depend on two composite elastic constants α and β in addition to p_1, p_2 of the two materials. When the two materials are interchanged, α, β, $\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$ change signs.

The elastic constants s'_{11}, s'_{12} and the eigenvalues p_1, p_2, which appeared in eqns (31) and (32) in general depend on the choice of coordinate system (x_1, x_2) in the bimaterial. When the coordinate system is rotated about the x_3-axis an angle θ, s'_{11}, s'_{12}, p_1 and p_2 referred to the rotated coordinate system change their values. Therefore α and β defined in eqn (37) are not invariant with θ unless υ and ω are. **S** and **L** are tensors of rank two when the transformation of the coordinate system is a rotation about the x_3-axis (Ting, 1982). Consequently, $\mathbf{SL}^{-1}$ is a tensor of rank two and its determinant, which is ω^2, is an invariant with θ. Thus ω is independent of the choice of coordinate system. It remains to fix the factor γ so that υ is also an invariant. Since $\mathbf{L}^{-1} = \upsilon\mathbf{P}$ and $\mathbf{L}^{-1}$ is a tensor, υ must be related to the invariants of $\mathbf{L}^{-1}$. The two principal invariants of $\mathbf{L}^{-1}$ are its determinant and its trace. From eqns $(29)_1$ and $(30)_1$,

$$|\mathbf{L}^{-1}| = \frac{(be-d^2)}{\gamma^2}\upsilon^2, \quad \operatorname{tr}\mathbf{L}^{-1} = \frac{(b+e)}{\gamma}\upsilon. \tag{39}$$

The coefficients of υ^2 and υ are $|\mathbf{P}|$ and $\operatorname{tr}\mathbf{P}$, respectively. We present below four choices of γ for which υ is an invariant.

(I) Let $\gamma = \frac{1}{2}(b+e)$. Then

$$|\mathbf{L}^{-1}| = h\upsilon^2, \quad \operatorname{tr}\mathbf{L}^{-1} = 2\upsilon, \quad |\mathbf{P}| = h \leqslant 1, \tag{40}$$

where

$$0 < h = \frac{4(be-d^2)}{(b+e)^2} = \frac{4be-4d^2}{4be+(b-e)^2} \leqslant 1. \tag{41}$$

Equations $(40)_2$ and $(40)_1$ tell us that υ and h are invariant. $h = 1$ when $b = e$ and $d = 0$, i.e. when $\mathbf{P}$ is a multiple of the identity matrix $\mathbf{I}$. It arises when $p_1 p_2 = -1$ or, by eqn (19), $s'_{22} = s'_{11}$ and $s'_{26} = -s'_{16}$.

(II) Let $\gamma = \sqrt{be-d^2}$. Then

$$|\mathbf{L}^{-1}| = \upsilon^2, \quad \operatorname{tr}\mathbf{L}^{-1} = \frac{2\upsilon}{\sqrt{h}}, \quad |\mathbf{P}| = 1, \tag{42}$$

and υ is an invariant.

If we rewrite eqn (32) as

$$\begin{aligned}
b &= \frac{1}{2i}[(p_1+p_2)-(\bar{p}_1+\bar{p}_2)] = \frac{1}{2i}[(p_1-\bar{p}_2)+(p_2-\bar{p}_1)] \\
d &= \frac{1}{2i}[p_1p_2-\bar{p}_1\bar{p}_2] = \frac{1}{2i}[p_2(p_1-\bar{p}_2)+\bar{p}_2(p_2-\bar{p}_1)] \\
e &= \frac{1}{2i}[p_1p_2(\bar{p}_1+\bar{p}_2)-\bar{p}_1\bar{p}_2(p_1+p_2)] = \frac{1}{2i}[\bar{p}_1p_2(p_1-\bar{p}_2)+p_1\bar{p}_2(p_2-\bar{p}_1)],
\end{aligned}$$

it can be shown easily that

$$\begin{aligned}
be-d^2 &= \frac{1}{4}(p_1-\bar{p}_1)(p_2-\bar{p}_2)(p_1-\bar{p}_2)(p_2-\bar{p}_1) > 0 \\
\tfrac{1}{2}(b+e) &= \frac{-i}{4}\{(p_1-\bar{p}_1)(p_2\bar{p}_2+1)+(p_2-\bar{p}_2)(p_1\bar{p}_1+1)\} > 0.
\end{aligned} \tag{43}$$

Thus, the choice of γ in cases (I) and (II) does not give a simple expression in terms of p_1, p_2. It has been proved in eqns (73) and (77) of Ting (1982) that

$$\frac{i(p\bar{p}+1)}{p-\bar{p}} \geqslant 1$$

is an invariant. The equality holds when $p = i$. Rewriting eqn $(43)_2$ as

$$\tfrac{1}{2}(b+e) = \frac{-1}{2}(p_1-\bar{p}_1)(p_2-\bar{p}_2)\frac{1}{\sqrt{g}} \tag{44}$$

where

$$\frac{1}{\sqrt{g}} = \frac{1}{2}\left\{\frac{i(p_1\bar{p}_1+1)}{p_1-\bar{p}_1} + \frac{i(p_2\bar{p}_2+1)}{p_2-\bar{p}_2}\right\} \geqslant 1,$$

g is an invariant. From $(43)_1$ and (44),

$$h = \frac{4(be-d^2)}{(b+e)^2} = \frac{(p_1-\bar{p}_2)(p_2-\bar{p}_1)}{(p_1-\bar{p}_1)(p_2-\bar{p}_2)}g.$$

Let p', p'' be the real and imaginary parts of p, respectively. We have

$$0 < \frac{g}{h} = \frac{(p_1-\bar{p}_1)(p_2-\bar{p}_2)}{(p_1-\bar{p}_2)(p_2-\bar{p}_1)} = \frac{(p_1''+p_2'')^2-(p_1''-p_2'')^2}{(p_1''+p_2'')^2+(p_1'-p_2')^2} \leqslant 1;$$

$g = h$ when $p_1 = p_2$. We now offer two more choices of γ.

(III) Let $\gamma = -\frac{1}{2}(p_1-\bar{p}_2)(p_2-\bar{p}_1)$. Then

$$|\mathbf{L}^{-1}| = \frac{g}{h}\upsilon^2, \quad \mathrm{tr}\,\mathbf{L}^{-1} = \frac{2\sqrt{g}}{h}\upsilon, \quad |\mathbf{P}| = \frac{g}{h} \leqslant 1, \tag{45}$$

and υ is an invariant.

(IV) Let $\gamma = -\frac{1}{2}(p_1-\bar{p}_1)(p_2-\bar{p}_2)$. Then

$$|\mathbf{L}^{-1}| = \frac{h}{g}\upsilon^2, \quad \mathrm{tr}\,\mathbf{L}^{-1} = \frac{2\upsilon}{\sqrt{g}}, \quad |\mathbf{P}| = \frac{h}{g} \geqslant 1, \tag{46}$$

and υ is an invariant.

There are, of course, other choices of γ for which υ is an invariant. When υ is an invariant, $\mathbf{P}$ is a tensor. The choice of γ in cases (II), (III) and (IV) are identical when $p_1 = p_2$. For the γ given by cases (I), (II), or (III), we have from eqns (40), (42) and (45)

$$|\mathbf{L}^{-1}| \leqslant \upsilon^2, \quad |\mathbf{P}| \leqslant 1. \tag{47}$$

As to the γ of case (IV), eqn (46) leads to eqn (47) with the inequality signs in eqn (47) reversed.

We will show that

$$\sqrt{|\mathbf{P}|} > s > 0, \quad s = \omega/\upsilon, \tag{48}$$

where s is an invariant. First, it can be verified by a direct calculation that

$$\mathbf{JPJP} = -|\mathbf{P}|\,\mathbf{I} = \mathbf{PJPJ} \tag{49}$$

for any 2×2 symmetric matrix $\mathbf{P}$. The tensor $\mathbf{S}$ deduced from eqns $(29)_{1,2}$ and (49) is

$$\mathbf{S} = s\mathbf{JP}^{-1} = s|\mathbf{P}|^{-1}\mathbf{PJ}.$$

Hence

$$\mathbf{S}^2 = -s^2|\mathbf{P}|^{-1}\mathbf{I}, \quad \{-\tfrac{1}{2}\mathrm{tr}\,(\mathbf{S}^2)\}^{1/2} = \frac{s}{\sqrt{|\mathbf{P}|}}.$$

Chadwick and Ting (1987) have proved for general anisotropic materials that

$$1 > \{-\tfrac{1}{2}\operatorname{tr}(\mathbf{S}^2)\}^{1/2} > 0.$$

Therefore eqn (48) holds. When γ is given by cases (I), (II) or (III), $|\mathbf{P}| \leqslant 1$ by eqn (47) and

$$1 > s > 0. \tag{50}$$

For isotropic materials $p_1 = p_2 = i$, and

$$s'_{11} = \frac{1-\nu}{2\mu}, \quad s'_{12} = \frac{-\nu}{2\mu},$$

where μ and ν are the shear modulus and Poisson ratio, respectively. Hence $d = 0$ and $b = e = 2$. All four choices of γ give the same γ. We have $\gamma = 2$, $|\mathbf{P}| = 1$ and

$$\mathbf{P} = \mathbf{I}, \quad \upsilon = \frac{1-\nu}{\mu}, \quad \omega = \frac{1-2\nu}{2\mu}.$$

Equation (38) simplifies to

$$\mathbf{D}^{-1}\mathbf{U} = \alpha\mathbf{I}, \quad \mathbf{D}^{-1}\mathbf{W} = \beta\mathbf{J}, \tag{51}$$

while eqn (37) takes the form

$$\alpha = \frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}, \quad \beta = \frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)} \tag{52}$$

in which $\kappa = 3-4\nu$. These are Dundurs constants for isotropic bimaterials under plane strain deformations.

Since υ_1, υ_2 are positive and nonzero,

$$-1 < \alpha < 1.$$

With eqns $(48)_2$ and $(37)_1$, β of eqn $(37)_2$ can be written as

$$\beta = \frac{\upsilon_1 s_1 - \upsilon_2 s_2}{\upsilon_1 + \upsilon_2} = \tfrac{1}{2}[(1+\alpha)s_1 - (1-\alpha)s_2].$$

Thus β is linear in α when s_1, s_2 are given. Let

$$\sqrt{|\mathbf{P}_i|} > (s_i)_{\max} \geqslant s_i \geqslant (s_i)_{\min} > 0 \quad (i = 1, 2).$$

In the (α, β)-plane, the point (α, β) for all possible combinations of elastic constants in the bimaterial is located inside the quadrilateral shown in Fig. 1. When γ is given by (I), (II),

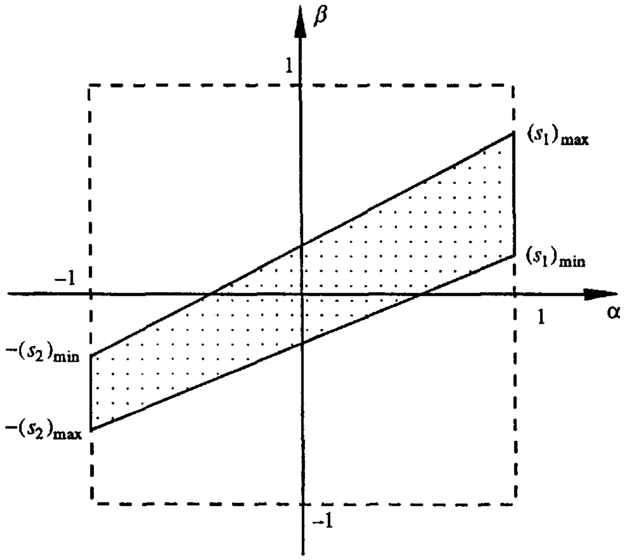

Fig. 1. The generalized Dundurs constants (α, β) fall inside the quadrilateral for all possible choices of elastic constants of monoclinic bimaterials. When γ is given by (I), (II) or (III), $(s_1)_{max} < 1$ and $(s_2)_{max} < 1$ so that the quadrilateral is bounded by a square of two units as shown. For other choices of γ, $(s_1)_{max}$ and $(s_2)_{max}$ can be larger than 1.

or (III) for which $|\mathbf{P}| \leqslant 1$, the quadrilateral is bounded by a square of two units. For isotropic materials (assuming that $0 \leqslant v < 1/2$),

$$s = \frac{1-2v}{2(1-v)}, \quad \tfrac{1}{2} \geqslant s_i > 0 \quad (i = 1, 2),$$

and the quadrilateral in Fig. 1 becomes a parallelogram (Dundurs, 1969a,b). It should be noted that for other choices of γ for which $|\mathbf{P}| > 1$, $(s_1)_{max}$ and $(s_2)_{max}$ can be very large and the quadrilateral in Fig. 1 extends to the outside of the square.

The expressions for the generalized Dundurs constants α, β given in eqn (37) are identical to those in eqn (5a) of Dundurs (1969a) for isotropic materials except a factor of four and different notations. The constants υ and ω have a simple physical interpretation when the materials are isotropic. For anisotropic materials consider a crack of length two units located at $x_2 = 0$, $-1 \leqslant x_1 \leqslant 1$ in an infinite monoclinic material. The crack surfaces are subjected to a uniform normal pressure of q. It can be shown from eqns (19) or (20) of Ting (1992b) that the displacement at the crack tips $x_1 = \pm 1$ are $u_1 = \mp \omega q$, $u_2 = 0$. Hence $2\omega q$ represents the shortening of the crack length. That ω is an invariant tells us that the shortening of the crack length is independent of the orientation of the crack in the material. As to υ, it is related to the determinant or the trace of $\mathbf{L}^{-1}$ which often appears in solutions to anisotropic elasticity problems.

The tensor $\mathbf{D}$ is positive definite, assuring us that the tensor $\mathbf{X}$ of eqn $(36)_1$, is also positive definite. Equation (49) applies to the tensor $\mathbf{X}$. Hence

$$\mathbf{X}^{-1} = -|\mathbf{X}|^{-1}\mathbf{JXJ}$$

and eqn (38) can be written more explicitly as

$$\begin{aligned} \mathbf{D}^{-1}\mathbf{U} &= \mathbf{X}^{-1}\{\alpha\mathbf{X} + (1-\alpha^2)(\mathbf{P}_1 - \mathbf{P}_2)\} \\ &= \alpha\mathbf{I} - (1-\alpha^2)|\mathbf{X}|^{-1}\mathbf{JXJ}(\mathbf{P}_1 - \mathbf{P}_2) \\ \mathbf{D}^{-1}\mathbf{W} &= 2\beta\mathbf{X}^{-1}\mathbf{J} = 2\beta|\mathbf{X}|^{-1}\mathbf{JX}. \end{aligned}$$

We also have

$$(\mathbf{D}^{-1}\mathbf{W})^2 = -4\beta^2|\mathbf{X}|^{-1}\mathbf{I}.$$

With $\mathbf{D}^{-1}$ being positive definite, $\mathbf{D}^{-1}\mathbf{U} = \mathbf{0}$ if and only if $\mathbf{U} = \mathbf{0}$ (or $\mathbf{L}_1 = \mathbf{L}_2$). Likewise, $\mathbf{D}^{-1}\mathbf{W} = \mathbf{0}$ if and only if $\mathbf{W} = \mathbf{0}$. By eqn (35), $\mathbf{W} = \mathbf{0}$ if and only if $\beta = 0$ (Ting, 1986; Qu and Bassani, 1989). Hence $\mathbf{D}^{-1}\mathbf{W} = \mathbf{0}$ and $\beta = 0$ are equivalent, and $\beta \neq 0$ for *mismatched* bimaterials.

From eqn (37), $\alpha = \beta = 0$ when υ and ω in the two materials are identical. For an isotropic bimaterial this implies that the two materials are identical. The same cannot be said of an anisotropic bimaterial. The two materials in an anisotropic bimaterial can be different even when $\alpha = \beta = 0$ and $\mathbf{L}_1 = \mathbf{L}_2$. They are identical when $\alpha = \beta = 0$ and p_1, p_2 in the two materials are the same.

6. IMPERFECTLY BONDED INTERFACE

For isotropic bimaterials the stress remains dependent on α and β even if the interface is not perfectly bonded (Dundurs, 1967b). We will show that the same is true for anisotropic bimaterials. The interface may consist of (a) interfacial cracks, (b) sliding interface without friction, and/or (c) sliding interface with friction. In all three cases the surface traction remains continuous across the interface so that eqn $(24)_2$ applies, i.e.

$$\mathbf{v}_1 + \mathbf{w}_1 = \boldsymbol{\phi} = \mathbf{v}_2 + \mathbf{w}_2 \tag{53}$$

where $\boldsymbol{\phi} = \boldsymbol{\phi}_1 = \boldsymbol{\phi}_2$ at the interface. For case (a), the $\boldsymbol{\phi}$ in eqn (53) is either a constant or zero, and no other boundary conditions are required. For cases (b) and (c), let $\mathbf{n}$ and $\mathbf{m}$ be the unit vectors normal and tangential to the interface, respectively. The continuity of the normal component of the displacement at the interface means that

$$\mathbf{n}^{\mathrm{T}}(\mathbf{u}_1 - \mathbf{u}_2) = 0. \tag{54}$$

This is the only equation that introduces elastic constants into the boundary conditions. The vanishing of the shear stress for case (b) and a relation between the normal stress and the shear stress due to friction for case (c) involve the stress function $\boldsymbol{\phi}$, which is independent of elastic constants other than p_1 and p_2. Following the derivation of eqn (25), we obtain, from eqns (53) and (54),

$$\mathbf{n}^{\mathrm{T}}[\mathbf{D}(\mathbf{v}_1 - \mathbf{v}_2) + \mathbf{U}(\mathbf{v}_2 - \mathbf{w}_1) - i\mathbf{W}(\mathbf{v}_1 + \mathbf{w}_1)] = \mathbf{0}$$

or, by eqns (33)–(35),

$$\mathbf{n}^{\mathrm{T}}[\mathbf{X}(\mathbf{v}_1 - \mathbf{v}_2) + \mathbf{Y}(\mathbf{v}_2 - \mathbf{w}_1) - 2i\beta\mathbf{J}(\mathbf{v}_1 + \mathbf{w}_1)] = \mathbf{0}. \tag{55}$$

$\mathbf{X}$ and $\mathbf{Y}$ depend on p_1, p_2 in the two materials and α. Therefore, the stress depends on the two generalized Dundurs constants α and β. The dependence on p_1 and p_2 is understood. From this result, the stress for collinear cracks (Hwu, 1993) and the Comninou crack (Comninou, 1977; Comninou and Dundurs, 1979; Wang and Choi, 1983; Wu and Hwang, 1990; Ni and Nemat-Nasser, 1991, 1992) depends on α and β only.

Dundurs (1975) has shown that, for isotropic bimaterials, the stress depends on α (not on β) if the interface is frictionless and is a straight line. Let the interface make an angle θ with the x_1-axis. We have

$$\mathbf{m}^{\mathrm{T}} = [\cos\theta, \sin\theta], \quad \mathbf{n}^{\mathrm{T}} = [-\sin\theta, \cos\theta]$$

where θ is a constant. Since $\mathbf{n}^{\mathrm{T}}\mathbf{J} = \mathbf{m}^{\mathrm{T}}$ eqn (55) reduces to

$$\mathbf{n}^{T}[\mathbf{X}(\mathbf{v}_1-\mathbf{v}_2)+\mathbf{Y}(\mathbf{v}_2-\mathbf{w}_1)]-2i\beta\mathbf{m}^{T}\boldsymbol{\phi} = 0, \tag{56}$$

use being made of eqn $(53)_1$. The vanishing of the shear stress at the frictionless surface means that $\mathbf{m}^T\boldsymbol{\phi}$ = constant at the interface. The stress function $\boldsymbol{\phi}$ is determined within an arbitrary constant. We may choose the arbitrary constant such that $\mathbf{m}^T\boldsymbol{\phi}$ vanishes. From eqn (56), the stress depends on α (not on β) for anisotropic bimaterials also.

7. SPECIAL MONOCLINIC BIMATERIALS

We have seen that the two 2×2 real tensors $\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$ in eqn (38) simplify to eqn (51) when the bimaterial is isotropic. There are special anisotropic bimaterials for which eqn (38) is simplified substantially. We begin with the case with few restrictions on elastic constants to the cases with more restrictions.

Case 1

Consider the special case in which the tensor $\mathbf{L}^{-1}$, and hence $\mathbf{P}$, is diagonal, i.e.

$$d = \mathrm{Im}\,(p_1 p_2) = 0. \tag{57a}$$

This implies that $p_1 = -\lambda\bar{p}_2$ where $\lambda > 0$ is an arbitrary constant. It was shown in eqn (5.1) of Ting (1992a) that eqn (57a) arises when

$$s'_{26}\sqrt{s'_{11}} = -s'_{16}\sqrt{s'_{22}}. \tag{57b}$$

Equation (57b) applies to both materials in the bimaterial. It imposes at most two restrictions on elastic constants of the bimaterial. Since eqn (57b) is automatically satisfied by isotropic materials and orthotropic materials with the symmetry planes coinciding with the coordinate planes (Dongye and Ting, 1989), it imposes no restriction on these materials. The matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are diagonal, so is $\mathbf{D}^{-1}\mathbf{U}$. The tensor $\mathbf{D}^{-1}\mathbf{W}$ is a product of a diagonal matrix by the constant matrix $\mathbf{J}$.

Case 2

Let the tensor $\mathbf{L}$ in the two materials be proportional, i.e.

$$\mathbf{L}_2 = \lambda\mathbf{L}_1 \quad \text{or} \quad \mathbf{L}_1^{-1} = \lambda\mathbf{L}_2^{-1}, \tag{58a}$$

where $\lambda > 0$ is an arbitrary constant. From eqn $(29)_1$ this implies that $\mathbf{P}_1$ and $\mathbf{P}_2$ are also proportional so that we write

$$\mathbf{P}_1 = \hat{\lambda}\mathbf{P}_2, \tag{58b}$$

where $\hat{\lambda} = (v_2/v_1)\lambda > 0$ is an arbitrary constant. Equation (58b) consists of three equations. It imposes at most two restrictions on elastic constants of the bimaterial but no restrictions for isotropic bimaterials. Equation (38) reduces to

$$\mathbf{D}^{-1}\mathbf{U} = \hat{\alpha}\mathbf{I}, \quad \mathbf{D}^{-1}\mathbf{W} = 2\hat{\beta}\mathbf{P}_2^{-1}\mathbf{J} \tag{59}$$

in which

$$\hat{\alpha} = \frac{(\hat{\lambda}-1)+\alpha(\hat{\lambda}+1)}{(\hat{\lambda}+1)+\alpha(\hat{\lambda}-1)}, \quad \hat{\beta} = \frac{\beta}{(\hat{\lambda}+1)+\alpha(\hat{\lambda}-1)}. \tag{60}$$

$\mathbf{D}^{-1}\mathbf{U}$ and $\mathbf{D}^{-1}\mathbf{W}$ depend on two composite elastic constants $\hat{\alpha}$ and $\hat{\beta}$ which are different from α and β in eqn (37). However $\hat{\alpha}$, $\hat{\beta}$ in eqn (60) and α, β in eqn (37) both reduce to the α, β in eqn (52) in the isotropic limit.

A more direct derivation of eqn (59) is to use eqn (58a) and write

$$\mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} = (\lambda+1)\mathbf{L}_2^{-1}, \quad \mathbf{U} = \mathbf{L}_1^{-1} - \mathbf{L}_2^{-1} = (\lambda-1)\mathbf{L}_2^{-1}.$$

Together with eqn $(35)_1$ one obtains eqn (59) where

$$\hat{\alpha} = \frac{(\lambda-1)}{(\lambda+1)}, \quad \hat{\beta} = \frac{(\omega_1-\omega_2)}{2\upsilon_2(\lambda+1)}. \tag{61}$$

This provides an alternate expression to eqn (60).

A special case of eqn (58a) is when $\lambda = 1$. By eqns $(61)_1$ and $(59)_1$, $\hat{\alpha}$ and $\mathbf{D}^{-1}\mathbf{U}$ vanish.

Case 3

A special case of eqn (58b) is when the eigenvalues of the two materials are identical, i.e.

$$p_1^{(1)} = p_1^{(2)}, \quad p_2^{(1)} = p_2^{(2)}. \tag{62}$$

Since $p_1^{(n)}, p_2^{(n)}$ are complex, eqn (62) imposes at most four restrictions on elastic constants of bimaterials, but no restrictions for isotropic bimaterials. It implies that

$$\mathbf{P}_1 = \mathbf{P}_2, \tag{63}$$

and eqn (38) simplifies to

$$\mathbf{D}^{-1}\mathbf{U} = \alpha\mathbf{I}, \quad \mathbf{D}^{-1}\mathbf{W} = \beta\mathbf{P}_2^{-1}\mathbf{J}, \tag{64}$$

in which α, β are given by eqn (37).

Case 4

Another special case of eqn (58) is when the elastic stiffnesses of the two materials are proportional, i.e.

$$C_{ij}^{(2)} = \lambda C_{ij}^{(1)} \quad \text{or} \quad s_{ij}^{(1)} = \lambda s_{ij}^{(2)}. \tag{65}$$

Since only six elastic constants each in the two materials are employed for plane strain deformations, eqn (65) imposes at most five restrictions on the twelve elastic constants of the bimaterial. It imposes only one restriction on isotropic bimaterials, namely, that the Poisson ratios of the two materials be identical. By eqn (19), the eigenvalues p_1, p_2 in the two materials are identical so that eqns (62)–(64) hold with α, β in eqn (64) given by

$$\alpha = \frac{(\lambda-1)}{(\lambda+1)}, \quad \beta = \alpha\frac{\omega_2}{\upsilon_2}. \tag{66}$$

8. MONOCLINIC BIMATERIALS WITH IDENTICAL p_1 AND p_2.

We will show in this section that every plane strain solution of an anisotropic elastic material or bimaterial is applicable to a wider class of materials or bimaterials. This is because there are elastic constants that are not employed in plane strain solutions. By reinstating these elastic constants which are arbitrary, one obtains a larger class of materials to which the original solution remains valid. We use the special materials classified as case 3 in the previous section as an illustration.

If two materials with the symmetry plane at $x_3 = 0$ have the identical eigenvalues p_1, p_2, by eqn (19) the ratios

$$\lambda_1 = \frac{s'_{16}}{s'_{11}}, \quad \lambda_2 = \frac{2s'_{12}+s'_{66}}{s'_{11}}, \quad \lambda_3 = \frac{s'_{26}}{s'_{11}}, \quad \lambda_4 = \frac{s'_{22}}{s'_{11}} \tag{67}$$

in the two materials must be identical. For isotropic materials these ratios are

$$\lambda_1 = \lambda_3 = 0, \quad \lambda_2 = 2, \quad \lambda_4 = 1 \tag{68}$$

so that eqn (67) imposes no restrictions on elastic constants of isotropic bimaterials. From eqn (3), the elastic stiffnesses of an anisotropic material that can produce plane strain deformations have the following structure:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}.$$

We have shown only the upper triangle because the matrix $\mathbf{C}$ is symmetric. Let $\mathbf{C}^{\circ}$ be the 5×5 matrix obtained from $\mathbf{C}$ by deleting the third row and the third column of $\mathbf{C}$. The 5×5 matrix s'_{ij} with $i \neq 3, j \neq 3$ is the inverse of $\mathbf{C}^{\circ}$ (Ting, 1992a). It can be shown that s'_{ij} has the structure

$$\mathbf{s}' = \begin{bmatrix} s'_{11} & s'_{12} & 0 & 0 & s'_{16} \\ & s'_{22} & 0 & 0 & s'_{26} \\ & & s'_{44} & s'_{45} & 0 \\ & & & s'_{55} & 0 \\ & & & & s'_{66} \end{bmatrix}.$$

Therefore the elastic compliances s_{ij} obtained from eqn (20) by taking s_{3j} as arbitrary have the structure

$$\mathbf{s} = \begin{bmatrix} s'_{11}+\gamma_1^2 & s'_{12}+\gamma_1\gamma_2 & \gamma_1\gamma_3 & \gamma_1\gamma_4 & \gamma_1\gamma_5 & \lambda_1 s'_{11}+\gamma_1\gamma_6 \\ & \lambda_4 s'_{11}+\gamma_2^2 & \gamma_2\gamma_3 & \gamma_2\gamma_4 & \gamma_2\gamma_5 & \lambda_3 s'_{11}+\gamma_2\gamma_6 \\ & & \gamma_3^2 & \gamma_3\gamma_4 & \gamma_3\gamma_5 & \gamma_3\gamma_6 \\ & & & s'_{44}+\gamma_4^2 & \gamma_4\gamma_5 & \gamma_4\gamma_6 \\ & & & & s'_{55}+\gamma_5^2 & \gamma_5\gamma_6 \\ & & & & & \lambda_2 s'_{11}-2s'_{12}+\gamma_6^2 \end{bmatrix}, \tag{69}$$

where use has been made of eqn (67) and $\gamma_i = s_{i3}/\sqrt{s_{33}}$ are arbitrary real constants with $\gamma_3 > 0$. Except λ_i $(i = 1, 2, 3, 4)$ that must be the same for the two materials, all s'_{ij} and γ_i appearing in eqn (69) can be arbitrary in each of the two materials. Anisotropic elastic materials represented by eqn (69) may possess no material symmetry planes at all (Cowin and Mehrabadi, 1987; Cowin, 1989; Ting, 1994a). Nevertheless, the eigenvalues p_1 and p_2 are identical in the two materials as long as λ_i $(i = 1, 2, 3, 4)$ are identical in the two materials. The solution for the stress in the bimaterial depends on p_1, p_2 and the generalized Dundurs constants α, β given in eqn (37), provided conditions (ii) and (iii) are satisfied.

If s'_{ij} in eqn (69) are specialized for isotropic materials and use is made of eqn (68), we obtain (Ting, 1994a)

$$\mathbf{s}=\frac{1}{2\mu}\begin{bmatrix} 1-\nu+\varepsilon_1^2 & -\nu+\varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \varepsilon_1\varepsilon_4 & \varepsilon_1\varepsilon_5 & \varepsilon_1\varepsilon_6 \\ & 1-\nu+\varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_4 & \varepsilon_2\varepsilon_5 & \varepsilon_2\varepsilon_6 \\ & & \varepsilon_3^2 & \varepsilon_3\varepsilon_4 & \varepsilon_3\varepsilon_5 & \varepsilon_3\varepsilon_6 \\ & & & 2+\varepsilon_4^2 & \varepsilon_4\varepsilon_5 & \varepsilon_4\varepsilon_6 \\ & & & & 2+\varepsilon_5^2 & \varepsilon_5\varepsilon_6 \\ & & & & & 2+\varepsilon_6^2 \end{bmatrix}, \tag{70}$$

where $\varepsilon_i = \sqrt{2\mu}\gamma_i$ $(i = 1,2,\ldots,6)$ are arbitrary constants with $\varepsilon_3 > 0$. Anisotropic materials represented by eqn (70) may possess no material symmetry. Nevertheless, the plane strain solutions for these materials are identical to plane strain solutions for isotropic materials. If the elastic constants of the two materials in the bimaterial are given by eqn (70) with arbitrary μ, ν and ε_i, the solution, and hence Dundurs constants, are identical to those for isotropic bimaterials.

9. PLANE STRESS DEFORMATIONS

When a monoclinic material with the symmetry plane at $x_3 = 0$ is under a plane stress deformation, the stress–strain laws and the equations of equilibrium are approximately identical to (1) and (2) for plane strain deformations if we replace C_{ij} by (Ting, 1994b)

$$C_{ij}^* = C_{ij} - \frac{C_{i3}C_{3j}}{C_{33}}.$$

This can be regarded as the 5×5 matrix C_{ij}^{o*} since the third row and the third column contain only zero elements. The reduced elastic compliances s'_{ij} are replaced by s'^*_{ij}, which is the inverse of $C_{ij}^{o}*$ (Ting, 1992a). It can be shown that s'^*_{ij} is the 5×5 matrix obtained from s_{ij} by deleting the third row and the third column, i.e.

$$s'^*_{ij} = s_{ij} - \delta_{i3}s_{3j} - s_{i3}\delta_{3j} + \delta_{i3}s_{33}\delta_{3j}.$$

Indeed, by virtue of the relation $C_{ij}s_{jk} = \delta_{ik}$, it is readily shown that

$$C_{ij}^* s'^*_{jk} = \delta_{ik} - \delta_{i3}\delta_{3k},$$

in which the right-hand side is the identity matrix when the third row and the third column are deleted. Therefore,

$$s'^*_{ij} = s_{ij}, \quad i \neq 3, \quad j \neq 3.$$

Thus, when s'_{ij} are replaced by s_{ij}, the results obtained in this paper apply to monoclinic materials and bimaterials under plane stress deformations.

For isotropic materials $p_1 = p_2 = i$,

$$s_{11} = \frac{1}{2\mu(1+\nu)}, \quad s_{12} = \frac{-\nu}{2\mu(1+\nu)},$$

so that

$$\upsilon = \frac{1}{\mu(1+\nu)}, \quad \omega = \frac{1-\nu}{2\mu(1+\nu)}.$$

Equation (37) again reduces to eqn (52) where

$$\kappa = \frac{3-\nu}{1+\nu}$$

for plane stress deformations.

10. CONCLUDING REMARKS

We have shown that the parameter ω is an invariant. With a proper choice of γ, the parameter υ is also an invariant, so are the generalized Dundurs constants α and β. If the invariance of α and β is of no concern, one may choose γ as any constant. The significance of the invariance of α, β is more than the independence on the choice of coordinate system. If we rotate material 1 about the x_3-axis before it is cut to the desired shape and is bonded to material 2, we would obtain a new anisotropic bimaterial. Since υ_1 and ω_1 are invariant with the rotation, α and β remain the same for the new bimaterial. Thus, when α and β are invariant, the generalized Dundurs constants do not depend on the individual orientation of the two materials in the bimaterial. In particular, $\alpha = \beta = 0$ if the two materials are identical but have different orientations.

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REFERENCES

Barnett, D. M. and Lothe, J. (1973). Synthesis of the sextic and the integral formalism for dislocations, Greens function and surface waves in anisotropic elastic solids. *Phys. Norv.* **7**, 13–19.
Chadwick, P. and Smith, G. D. (1977). Foundations of the theory of surface waves in anisotropic elastic materials. *Adv. Appl. Mech.* **17**, 401–418.
Chadwick, P. and Ting, T. C. T. (1987). On the structure and invariance of the Barnett–Lothe tensors. *Q. Appl. Math.* **45**, 419–427.
Comninou, M. (1977). The interface crack. *J. Appl. Mech.* **44**, 631–636.
Comninou, M. and Dundurs, J. (1979). On the frictional contact in crack analysis. *Engng Fract. Mech.* **12**, 117–123.
Cowin, S. C. (1989). Properties of the anisotropic elasticity tensor. *Q. J. Mech. Appl. Math.* **42**, 249–266.
Cowin, S. C. and Mehrabadi, M. M. (1987). On the identification of material symmetry for anisotropic elastic materials. *Q. J. Mech. Appl. Math.* **40**, 451–476.
Dongye, C. and Ting, T. C. T. (1989). Explicit expressions of Barnett–Lothe tensors and their associated tensors for orthotropic materials. *Q. Appl. Math.* **47**, 723–734.
Dundurs, J. (1967a). Dependence of stress on Poisson's ratio in plane elasticity. *Int. J. Solids Structures* **3**, 1013–1021.
Dundurs J. (1967b). Effect of elastic constants on stress in a composite under plane deformation. *J. Composite Mater.* **1**, 310–322.
Dundurs, J. (1969a). Discussion of a paper by D. B. Bogy. *J. Appl. Mech.* **36**, 650–652.
Dundurs, J. (1969b). Elastic interaction of dislocations with inhomogeneities. In *Mathematical Theory of Dislocations* (Edited by T. Mura), pp. 70–115. ASME, New York.
Dundurs, J. (1970). Some properties of elastic stresses in a composite. In *Recent Advances in Engineering* (Edited by A. C. Eringen), Vol. 5, pp. 203–216. Gordon and Breach, NY.
Dundurs, J. (1975). Properties of elastic bodies in contact. In *The Mechanics of the Contact between Deformable Bodies* (Edited by A. D. de Pater and J. J. Kalker) pp. 54–66, Delft University Press.
Eshelby, J. D., Read, W. T. and Shockley, W. (1953). Anisotropic elasticity with applications to dislocation theory. *Acta Metall.* **1**, 251–259.
Horgan, C. O. and Miller, K. L. (1994). Antiplane shear deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids. *J. Appl. Mech.* **61**, 23–29.
Hwu, C. (1993). Explicit solutions for the collinear interface crack problems. *Int. J. Solids Structures* **30**, 301–312.
Ingebrigtsen, K. A. and Tonning, A. (1969). Elastic surface waves in crystals. *Phys. Rev.* **184**, 942–951.
Lekhnitskii, S. G. (1950). *Theory of Elasticity of an Anisotropic Body*. Gostekhizdat, Moscow (in Russian). Theory of Elasticity of an Anisotropic Elastic Body (1963). Holden-Day, San Francisco (in English).
Michell, J. H. (1899). On the direct determination of stress in an elastic solid, with application to the theory of plates. *Proc. London Math. Soc.* **31**, 100.
Ni, L. and Nemat-Nasser, S. (1991). Interface cracks in anisotropic dissimilar materials: an analytic solution. *J. Mech. Phys. Solids* **39**, 113–144.
Ni, L. and Nemat-Nasser, S. (1992). Interface cracks in anisotropic dissimilar materials: general case. *Q. Appl. Math.* **50**, 305–322.
Qu, J. and Bassani, J. L. (1989). Cracks on bimaterial and bicrystal interfaces. *J. Mech. Phys. Solids* **37**, 417–433.

Qu, J. and Li, Q. (1991). Interfacial dislocation and its application to interface crack in anisotropic bimaterials. *J. Elasticity* **26,** 167–195.

Stroh, A. N. (1958). Dislocations and cracks in anisotropic elasticity. *Phil. Mag.* **3,** 625–646.

Stroh, A. N. (1962). Steady state problems in anisotropic elasticity. *J. Math. Phys.* **41,** 77–103.

Suo, Z (1990). Singularities, interfaces and cracks in dissimilar anisotropic media. *Proc. R. Soc. London* **A427,** 331–358.

Ting, T. C. T. (1982). Effects of change of reference coordinate system on the stress analyses of anisotropic materials. *Int. J. Solids Structures* **18,** 139–152.

Ting, T. C. T. (1986). Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. *Int. J. Solids Structures* **22,** 965–983.

Ting, T. C. T. (1992a). Barnett–Lothe tensors and their associated tensors for monoclinic materials with the symmetry plane at $x_3 = 0$. *J. Elasticity* **27,** 143–165.

Ting, T. C. T. (1992b). Interface cracks in anisotropic elastic bimaterials—a decomposition principle. *Int. J. Solids Structures* **29,** 1989–2003.

Ting, T. C. T. (1994a). On anisotropic elastic materials that possess three identical Stroh eigenvalues as do isotropic materials. *Q. Appl. Math.* **52,** 363–375.

Ting, T. C. T. (1994b). Antiplane deformations of anisotropic elastic materials. In *Recent Advances in Elasticity, Inelasticity and Viscoelasticity* (Edited by K. R. Rajagopal). World Scientific, New Jersey.

Wang, S. S. and Choi, I. (1982). Boundary-layer effects in composite laminates. Part I—free-edge stress singularities. *J. Appl. Mech.* **49,** 541–548.

Wang, S. S. and Choi, I. (1983). The interface crack between dissimilar anisotropic composite materials. *J. Appl. Mech.* **50,** 169–178.

Wu, K. C. and Hwang, S. J. (1990). Correspondence relations for the interface crack in monoclinic composites under mixed loading. *J. Appl. Mech.* **57,** 894–900.